Homework 7 Solutions

Math 131B-2

- For each of these, we apply the Root Test and then examine the endpoints of the interval individually. The intervals of convergence are $[4.5, 5.5]$, $(0, 2)$, $(-6, 4]$, and $\{3\}$.

- Let $s_n(x) = 1 + x + x^2 + \cdots + x^n$. Then on $(-1, 1)$, s_n is bounded, since $|s_n(x)| < n+1$. We know the s_n converge pointwise to $f(x) = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$. If the convergence were uniform on $(-1, 1)$, then we would expect the limit function $f(x)$ to be bounded on $(-1, 1)$ as well, but it clearly isn't.

- (9.33) (a) Consider the first derivative $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$. This limit exists if and only if the corresponding limits from the right and left exist. Now the limit from the left is $\lim_{x\to 0^-} \frac{0}{x} = 0$ and the limit from the right is

$$\begin{aligned}\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x} &= \lim_{t\to\infty} \frac{t}{e^{t^2}} \\ &= \lim_{t\to\infty} \frac{1}{2te^{t^2}} \\ &= 0\end{aligned}$$

Here in the first step we have used the replacement $t = \frac{1}{x}$, and in the second we have used L'Hospital's Rule, since $t, e^{t^2} \to \infty$ as $t \to \infty$. Therefore $f'(0) = 0$, and in general $f'(x) = \frac{-2}{x^3} e^{-\frac{1}{x^2}}$ for $x > 0$ and $f'(x) = 0$ for $x \leq 0$.

Subsequent derivatives are similar: for every n, the function $f^(n)(x)$ is 0 when $x \leq 0$ and a sum of terms of the form $\frac{ce^{-\frac{1}{x^2}}}{x^k}$ for some k when $x > 0$, so an argument similar to the one above, with repeated applications of L'Hospital's Rule, shows that $f^{(n+1)}(0) = 0$.

(b) By part (a), the Taylor series of f is identically 0. This certainly converges, but since $f(x) > 0$ for $x > 0$, there is no interval on which the Taylor series represents f.

- (9.36) Since all a_n are nonnegative, $\sum a_n$ must diverge to ∞. Let $M > 0$. Then there is some N such that for $n \geq N$, the partial sum $\sum_{n=0} N a_n > 2M$. Moreover, there is some δ such that $x \in (1-\delta, 1)$ implies that $x^N > \frac{1}{2}$. Then for all $x \in (1-\delta, 1)$, we have $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0} N a_n x^n + \sum_{n=N+1}^{\infty} a_n x^n \geq \sum_{n=0}^{N} a_n x^n \geq \sum_{n=0}^{N} a_n \cdot \frac{1}{2} \geq M$. So $\lim_{x\to 1^-} a_n x^n = \infty$.

- (9.37) By 9.36, if $\sum a_n$ were to diverge, then $\sum a_n x^n$ would go to infinity as $x \to 1^-$. As this is not the case, we must have $\sum a_n$ convergent. So by Abel's theorem, since $\sum a_n$ exists, we must have $\sum a_n = \lim_{x\to 1^-} \sum a_n x^n = A$.

- (9.21) When $x \neq 1$, notice that the series is the integral of $\sum_{n=0}^{\infty} x^{2n} - \frac{1}{2}\sum_{n=0}^{\infty} x^n$, which converges to $\frac{1}{1-x^2} - \frac{1}{2(1-x)} = \frac{1}{2(1+x)}$ on $(-1,1)$. Ergo the original series converges to $\frac{1}{2}\ln(1+x)$ on $[0,1)$. When $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ the alternating harmonic series, which converges to $\ln(2)$, not $\frac{1}{2}\ln(2)$. Therefore the series converges pointwise, but the limit cannot be continuous at $x = 1$, and therefore we conclude that the convergence is not uniform. Why is this not a contradiction to Abel's Theorem? Because the series is not actually a power series as written. We would need to rearrange the terms of the series to have a power series $\sum c_n x^n$, and in doing so we would change the conditional convergence at the endpoint.

- Recall that $B(M \to S)$ (or any of the other notations mentioned in class) is the set of bounded functions $f\colon M \to s$ and $C(M \to S) \subset B(M \to S)$ is the subspace consisting of bounded continuous functions.

  - We check the axioms of a metric space. First, if $f\colon M \to S$, notice that $d_\infty(f,f) = \sup_{x\in M} d_S(f(x), f(x)) = 0$. If $f, g\colon M \to S$ such that $f \neq g$, there must be some $x_0 \in M$ such that $f(x_0) \neq g(x_0)$, ergo $d_\infty(f,g) = \sup_{x\in M} d_S(f(x), g(x)) \geq d_S(f(x_0), g(x_0)) > 0$. Second, $d_\infty(f,g) = d_\infty(g,f)$ because d_S is symmetric.

    The interesting axiom is the triangle inequality. Let $f, g, h\colon M \to S$ be bounded, and let $a = d_\infty(f,g) = \sup_{x\in M} d_S(f(x), g(x))$. Then given $\epsilon > 0$, there is some $x_0 \in S$ such that $d_S(f(x_0), g(x_0)) > a - \epsilon$. Therefore

$$\begin{aligned} a - \epsilon &< d_S(f(x_0), g(x_0) \\ &\leq d_S(f(x_0), h(x_0)) + d_S(h(x_0), g(x_0)) \\ &\leq \sup_{x\in M} d_S(f(x), h(x)) + \sup_{x\in M} d_S((h(x), g(x)) \\ &= d_\infty(f,h) + d_\infty(h,g) \end{aligned}$$

    Ergo $d_\infty(f,g) - \epsilon < d_\infty(f,h) + d_\infty(h,g)$ for all $\epsilon > 0$. This implies that in fact $d_\infty(f,g) - \epsilon \leq d_\infty(f,h) + d_\infty(h,g)$.

  - Let $M = [0,1]$ and $S = \mathbb{R}$. Then if $f(x) = 0$, $g(x) = M$ for $M > 0$, $d_\infty(f,g) = M$. Therefore $C(M \to S)$ admits arbitrarily large distances and is not bounded. Since all compact sets are bounded, $C(M \to S)$ is not compact.

  - Let $M = \{0\}$ and $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let $f_n(0) = \frac{1}{n}$, which is trivially bounded and continuous. Then $d_\infty(f_n, f_m) = |\frac{1}{n} - \frac{1}{m}|$, so the sequence $\{f_n\}$ is Cauchy. However, it does not converge to any $f : M \to S$.

- *Double sums* (a) Recall that a series of nonnegative terms b_i either converges or diverges to ∞, and in either case the expression $\sum b_i$ makes sense. Similarly, $\sum_i \sum_j a_{ij}$ either converges or diverges to ∞ if $a_{ij} \geq 0$. If $\sum_i \sum_j a_{ij}$ converges, then since $|a_{ij}| = a_i j$, by the theorem proved in class, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$. However, the opposite direction is also true, so if either of $\sum_i \sum_j a_{ij}$ or $\sum_j \sum_i a_{ij}$ is finite they are equal, and otherwise they are both $+\infty$.

  (b) If we switch the order of summation, we see that

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(i+1)!} \left(\frac{i}{i+1} \right)^j = \sum_{i=0}^{\infty} \frac{1}{(i+1)!}(i+1)$$
$$= \sum_{i=0}^{\infty} \frac{1}{i!}$$
$$= e$$

  . By part (a), this is also $\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij}$.

- *Approximating integrals* (a) $\int_0^1 \cos(x^2) = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}$.

  (b) We see that $\int_0^1 \cos^2(x) \approx 1 - \frac{1}{10} + \frac{1}{216}$, with error less than the absolute value of the next term of the series, $\frac{1}{9360}$. Our answer is an overestimate because we ended on a positive term.